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Path calculus for Z2–Higgs lattice gauge model

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Abstract. We propose a non-perturbative approach to the Z2 model on the lattice. A Hamiltonian calculation in $d=2$ is performed using a loop-labelled basis explicitly constructed with 'cluster' structures. The Schrödinger problem leads to a linear set of finite difference equations that is solved numerically. Solutions for the energy of the vacuum and first excitation mass gap are found.

1. Introduction

The Z2 gauge–Higgs model has attracted considerable attention [1–9] because it exhibits an interesting phase structure and yet it is sufficiently simple to greatly simplify its analysis. It is currently of interest to particle theorists in the study of confinement and Higgs phenomena. A qualitative structure of the phase diagram has been described by Fradkin and Shenker [10], among others [1–9]. This model has been studied within the Euclidean path integral formalism [5, 6], which is amenable to numerical implementations, but is probably not the most economical device to understand the real dynamics of the system. The large number of irrelevant degrees of freedom constitutes a significant difficulty to the numerical approach. A natural step to identify the real degrees of freedom of the model is the Hamiltonian formalism of Kogut and Susskind [11]. However, even in this formalism, one still has to deal with a spatial gauge symmetry. This degeneration can be eliminated if we work in the subspace defined by Gauss's law (gauge-invariant sector) [3–4, 12–14].

Two of the authors have proposed [12] an analytical Hamiltonian approach to continuum gauge theories and to their lattice version. The method is based in the formulation in a space of states labelled by elements of the group of loops [12–16] which parametrize the subspace of solutions of Gauss's constraints. The first application of this approach to matter fields was the formulation of the dynamics of an SU(2) gauge theory with Higgs bosons [14] using open paths instead of closed ones or loops. Recently, another gauge theory with matter fields, the compact QED with fermionic matter fields, has been studied within this framework [17].

Recently [13], a general computational method based on loop techniques was presented to deal with the Hamiltonian SU(2) lattice gauge model, and the SU(3) theory was also studied in [18]. Owing to the fact that linked clusters played a relevant role in the construction of the basis, the method can be classified as a strong-coupling

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series approach with some built-in mechanism to extrapolate towards the weak-coupling region.

The structure of this paper is as follows. In section 2 we introduce the path representation for the Z2 gauge-Higgs model. In section 3 we propose a cluster approximation which reduces the Schrödinger equation to a partial difference equation. We compute the ground state energy and the mass gap in order to obtain information about the phase diagram. In section 4 we discuss our results and in section 5 we make some final comments.

2. The *P* representation

The Z2 gauge-Higgs Hamiltonian is given by

$$H = -\frac{1}{\lambda\sqrt{\mu}}\sum_p \varphi(p) - \rho\sqrt{\mu}\sum_l \varphi(l) + \frac{\lambda}{\sqrt{\mu}}\sum_l [1-(\sigma_1)_l] + \frac{\sqrt{\mu}}{\rho}\sum_x [1-(\tau_1)_x] \tag{2.1}$$

where p and l label plaquettes and links respectively. We define $\varphi(p) \equiv (\sigma_3\sigma_3\sigma_3\sigma_3)_p$ and $\varphi(l) \equiv (\tau_3\sigma_3\tau_3)_l$, where the σs and τs are usual Pauli matrices associated with links and vertices respectively. The parameters λ and ρ measure the gauge and Higgs coupling. The parameter μ, apparently, is irrelevant near the critical point [7]. However, in [4] it is shown that the phase diagram is quantitatively sensitive to the choice of μ. When $\mu = 1$, the symmetry of the model simplifies the numerical calculus. From now on we choose $\mu = 1$.

The configuration basis $|s, u\rangle$, labelled by the values of the spin on each site s_x and each link u_l, is not gauge-invariant. A description of states on a gauge-invariant basis may be given in terms of the dual basis $|P\rangle$ defined by

$$\langle s, u|P\rangle = 2^{-Q/2}2^{-V/2}\left(\prod_{l\in C} u_l\right)\left[\prod_{i=1}^{n} s_{x_i}\left(\prod_{l\in P_i} u_l\right)s_{y_i}\right] \tag{2.2}$$

where Q is the number of links and V the number of sites of the lattice, C is an 'unoriented' spatial closed path and P_i are open spatial connected 'unoriented' paths with end points x_i and y_i. From (2.2) it is easy to see that 'double' links and 'double' sites do not affect the scalar product for the P-representation in the Z2 gauge-Higgs case.

In the P-representation, the Hamiltonian operators act as follows:

$$\begin{aligned} &\varphi(p)|P\rangle = |pP\rangle \qquad \varphi(l)|P\rangle = |lP\rangle \\ &(\sigma_1)_l|P\rangle = (1-2\delta_{l,P})|P\rangle \qquad (\tau_1)_x|P\rangle = (1-2\delta_{x,P})|P\rangle \end{aligned} \tag{2.3}$$

where $\delta_{l,P} = 1$ if link l belongs to P and 0 otherwise, while $\delta_{x,P}$ measures if the site x belongs to P. Thus, the Hamiltonian may be rewritten as

$$H = -\frac{1}{\lambda}\sum_p \varphi(p) - \rho\sum_l \varphi(l) + 2\lambda L_{\text{op}} + \frac{4}{\rho}N_{\text{op}} \tag{2.4}$$

where the operator L_{op} measures the length of P and N_{op} the number of connected pieces of P.

3. The cluster approximation

To solve the Schrödinger equation we use a cluster approximation [13] that holds at the strong-coupling region. In this approximation we will consider a basis of states

labelled by an unordered list of connected paths (open or closed) corresponding to the Z2 gauge-Higgs case. A given list may contain several sublists located wide apart in the lattice that we will call clusters [13]. In order to describe the ground state of the system we think of a cluster as a class of equivalent paths differing by Poincaré transformations of the lattice. Within this view, it is only necessary to specify the occupation number of each non-equivalent cluster. Therefore, by taking an arbitrary numbering of clusters, a general list will be specified as

$$|n_1, n_2, \ldots, n_F, \ldots\rangle \tag{3.1}$$

where n_i denotes the number of times the cluster i appears in the list. The next step is to introduce an ordering among clusters. This may be done in a recursive way. The null path is the order zero while a single plaquette and a single link is the order one. The Nth order is obtained by the action of the Hamiltonian on clusters of order $N-1$, keeping only clusters obtained when the plaquettes are appended in direct contact with a link of a closed path and when links are appended in direct contact with a link or an extreme of an open path.

Now we propose an approximation procedure where we consider only clusters of order less than or equal to two. There are five clusters to this order in a $d=2$ lattice, the single plaquette, single link, double plaquette, double links in the same direction and double links in transversal direction. Accordingly, the basis is restricted to states

$$|n_1, n_2, n_3, n_4, n_5\rangle. \tag{3.2}$$

The following step is to evaluate the action of the Hamiltonian on the basis given by (3.2). For convenience we work with $H' \equiv \lambda H$, where H is defined in (2.4). There will be transitions to clusters that are considered to the order two and others that produce new clusters that are outside the range of the list considered in (3.2); the latter cannot be processed exactly. In order to improve the method and induce a good extrapolation to the weak-coupling region we introduce what we call collective variables [13]. These variables are additive quantities that take values on every cluster. Their use allows the transition to a cluster outside of the order to be counted partially by keeping track of the value of the collective quantity for the reached cluster.

The collective variables we use are the length L and the number of connected pieces N of the path. For the state in the number representation (3.2), these variables take the values

$$L = 4n_1 + n_2 + 6n_3 + 2n_4 + 2n_5 \qquad N = n_2 + n_4 + n_5. \tag{3.3}$$

Now it is possible to exchange some variables, say n_1 and n_2, by a linear combination of L and N, and consider states of the form

$$\left|\frac{L - N - 6n_3 - n_4 - n_5}{4}, N - n_4 - n_5, n_3, n_4, n_5\right\rangle \tag{3.4}$$

so when we evaluate a transition induced by H' we must take into account the changes in L and N as those in n_3, n_4 and n_5, in order to evaluate the changes of n_1 and n_2.

Under these assumptions the Schrödinger equation

$$\langle\Psi_0|H'|n_1, n_2, n_3, n_4, n_5\rangle = E_0'\langle\Psi_0|n_1, n_2, n_3, n_4, n_5\rangle \tag{3.5}$$

leads to a finite difference equation:

$$
\begin{aligned}
n_1[&(5\Psi_0(1,0,0,0,0)-4\Psi_0(-1,0,1,0,0)-\Psi_0(-1,0,0,0,0))\\
&+4\rho\lambda(\Psi_0(0,1,0,0,0)-\Psi_0(-\tfrac{1}{2},1,0,0,0))+8\lambda^2\Psi_0(0,0,0,0,0)]\\
&+n_2[2(\Psi_0(1,0,0,0,0)-\Psi_0(\tfrac{1}{2},0,0,0,0))\\
&-\rho\lambda(\Psi_0(0,-1,0,0,0)+2\Psi_0(0,-1,0,1,0)+4\Psi_0(0,-1,0,0,1))\\
&+7\rho\lambda\Psi_0(0,1,0,0,0)+\lambda(2\lambda+4/\rho)\Psi_0(0,0,0,0,0)]\\
&+n_3[8\Psi_0(1,0,0,0,0)-2\Psi_0(1,0,-1,0,0)-6\Psi_0(0,1,-1,0,0)\\
&+6\rho\lambda(\Psi_0(0,1,0,0,0)-\Psi_0(1,1,-1,0,0))+12\lambda^2\Psi_0(0,0,0,0,0)]\\
&+n_4[4(\Psi_0(1,0,0,0,0)-\Psi_0(\tfrac{3}{4},1,0,-1,0))\\
&+\rho\lambda(8\Psi_0(0,1,0,0,0)-2\Psi_0(0,1,0,-1,0)-6\Psi_0(\tfrac{1}{2},1,0,-1,0))\\
&+4\lambda(\lambda+1/\rho)\Psi_0(0,0,0,0,0)]\\
&+n_5[3\Psi_0(1,0,0,0,0)-\Psi_0(0,0,0,0,0)-2\Psi_0(\tfrac{3}{4},1,0,0,-1)\\
&-2\rho\lambda(\Psi_0(0,1,0,0,-1)+3\Psi_0(\tfrac{1}{2},1,0,0,-1)-4\Psi_0(0,1,0,0,0))\\
&+4\lambda(\lambda+1/\rho)\Psi_0(0,0,0,0,0)]\\
&=M(\Psi_0(1,0,0,0,0)+2\rho\lambda\Psi_0(0,1,0,0,0)-\varepsilon_0\Psi_0(0,0,0,0,0))
\end{aligned}
\tag{3.6}
$$

where for conciseness we write in the arguments of the vacuum wave function, Ψ_0, only the increments or decrements of the variables n_1, n_2, n_3, n_4 and n_5 respectively. M is the number of plaquettes of the lattice and ε_0 is the vacuum energy density defined by

$$\varepsilon_0 \equiv -\frac{E_0'}{M}. \tag{3.7}$$

Equation (3.6) has solutions of the form

$$\Psi_0(n_1, n_2, n_3, n_4, n_5) = x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4}x_5^{n_5}. \tag{3.8}$$

The term proportional to M in (3.6) must vanish separately and produces the dispersion relation

$$\varepsilon_0 = x_1 + 2\rho\lambda x_2 \tag{3.9}$$

while the terms proportional to n_i vanishing separately produce a nonlinear system of equations:

$$f_i(x_1, x_2, x_3, x_4, x_5) = 0 \tag{3.10}$$

where $i = 1, \ldots, 5$. From (3.10) all the five variables may be determined.

To find solutions that will describe the elementary excitations, we propose polynomic modulations of the ground state. We look for solutions of the form

$$
\begin{aligned}
&\Psi(n_1, n_2, n_3, n_4, n_5)\\
&= x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4}x_5^{n_5}\left(1+\frac{a_1n_1+a_2n_2+a_3n_3+a_4n_4+a_5n_5}{M}\right)
\end{aligned}
\tag{3.11}
$$

from which the mass gap Δ is obtained as the lowest eigenvalue of the proper value

problem

$$D_{ij}a_j = \Delta a_i \tag{3.12}$$

where

$$D_{ij} \equiv \left(x_j \frac{\partial f_i}{\partial x_j} \right)_{\text{vacuum}} \qquad \text{(without summation in } j\text{)}. \tag{3.13}$$

As in (3.9), we obtain the following relation for the a_is:

$$a_1 x_1 + 2\rho\lambda a_2 x_2 + \Delta = 0. \tag{3.14}$$

4. Results

In figure 1 we show the solution for ε_0 represented versus λ for diverse values of the parameter ρ. The introduction of collective variables allows us to reach the weak-coupling region. However, we do not have an exact description of this region since the method is an asymptotic strong-coupling series carried out only to the second order of approximation. Figure 2 shows the mass gap versus λ. In order to detect a phase

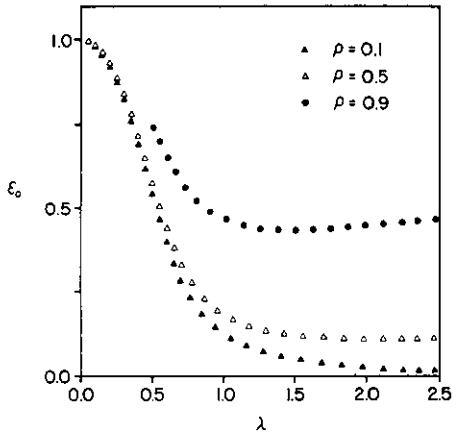

Figure 1. Vacuum energy versus λ for $\rho = 0.1$, 0.5 and 0.9.

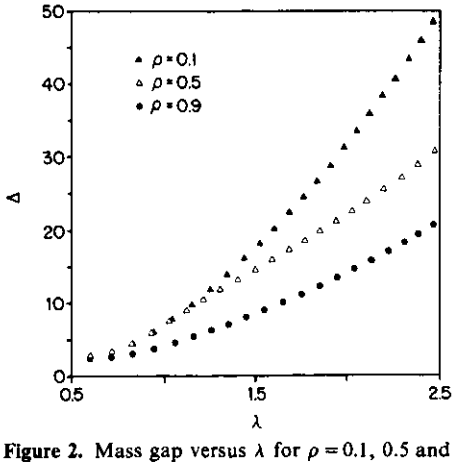

Figure 2. Mass gap versus λ for $\rho = 0.1$, 0.5 and 0.9.

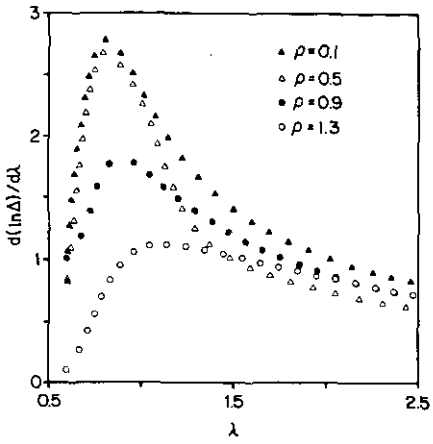

Figure 3. Logarithmic derivative of mass gap versus λ for $\rho = 0.1$, 0.5, 0.9 and 1.3. The value at the maximum is taken as an estimate of the transition point for each ρ.

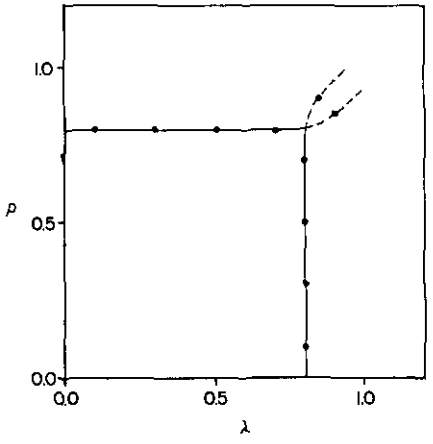

Figure 4. Phase diagram. The full lines indicate the estimated second-order transition.

Table 1. Critical points as predicted by several approaches. The last corresponds to the present approach.

λ_g	λ_c	
0.556	0.615	Finite-lattice method [4]
0.37	—	Strong-coupling expansion [7]
0.573	0.59	Strong-coupling expansion [8]
—	0.68	Variational method [3]
0.80	0.81	Clusters with collective variables

transition it is convenient to plot the logarithmic derivative of the mass gap versus λ (figure 3). In this graph the curves corresponding to diverse values of ρ exhibit a peak that dilutes when the value of ρ increases until it eventually disappears. The peak indicates a transition for every ρ at a certain value of λ and, as is expected [1-9], the transition disappears when one goes into the phase diagram. The transition that we can detect is a second-order one since we obtain it from a maximum in figure 3. Using these criteria and the duality properties of the Hamiltonian we can obtain a phase diagram (figure 4) where the full lines indicate the second-order transition. Our results for the value of λ_g at which the transition occurs for the pure gauge model and for the value $\rho_c = \lambda_c$ where the two second-order lines cross over are shown in table 1, where they are compared with those of Irving and Thomas [4], Banks and Sinclair [7] and Lamont [8].

5. Final comments

The geometric cluster approach that we propose provides an alternative calculation for a basically strong-coupling method since the introduction of collective variables accelerates the convergence toward the weak-coupling regions. This is an important feature in order to describe phase transitions.

Although the order of the approximation is low we obtain a description of the phase boundaries of the model. In order to improve the results for the transition points it is necessary to go to higher orders of the cluster approximation.

The advantage of the method is that the mechanisms for the generation of the clusters and the evaluation of the transitions induced by the Hamiltonian can be automatized by symbolic computer manipulation. In spite of the fact that the number of clusters grows with the order and this will complicate the calculus it is important to mention that an appropriate set of collective variables may accelerate the convergence toward other regions of the phase space. Efforts are being made in search of a method to select the relevant cluster variables that characterize the vacuum of the model [18]. This would greatly reduce the number of variables and this is important at higher orders.

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